

A RECURSION FOR DIVISOR FUNCTION OVER DIVISORS BELONGING TO A PRESCRIBED FINITE SEQUENCE OF POSITIVE INTEGERS AND A SOLUTION OF THE LAHIRI PROBLEM FOR DIVISOR FUNCTION

$$\sigma_x(n)$$

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ABSTRACT. For a finite sequence of positive integers $A = \{a_j\}_{j=1}^k$, we prove a recursion for divisor function $\sigma_x^{(A)}(n) = \sum_{d|n, d \in A} d^x$. As a corollary, we give an affirmative solution of the problem posed in 1969 by D. B. Lahiri [3]: to find an identity for divisor function $\sigma_x(n)$ similar to the classic pentagonal identity in case of $x = 1$.

1. INTRODUCTION AND MAIN RESULTS

We start with the two well known beautiful classical recursions. Let $p(n)$ be the number of all partitions of positive integer n and $\sigma(n)$ be the sum of its divisors. Then (sf [1],[5]) we have

$$(1) \quad p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \dots$$

$$(2) \quad \sigma(n) = \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \sigma(n-12) + \sigma(n-15) - \dots$$

where the numbers 1,2,5,7,12,15,... appearing in the successive terms in (1)-(2) are the positive pentagonal numbers $\{v_m\}$ given by

$$(3) \quad v_m = m(3m \mp 1)/2, \quad m = 1, 2, \dots$$

In identities (1)-(2) we accept that $p(m) = 0$, $\sigma(m) = 0$ when $m < 0$. The only formal difference is that (1) is true with the understanding that

$$(4) \quad p(0) = 1,$$

while (2) is valid with the understanding that

$$(5) \quad \sigma(0) = n.$$

Note that, formulas (1)-(2) are proved with help of the famous Euler pentagonal identity

$$(6) \quad \prod_{n=1}^{\infty} (1 - q^n) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2}.$$

In its turn, a combinatorial proof of (6) is based on the following statement (sf [1]). Let $p_e(n)$ ($p_o(n)$) denote the number of partitions of n into even (odd) number of distinct parts. Then

$$(7) \quad p_e(n) - p_o(n) = \begin{cases} (-1)^m, & \text{if } n = m(3m \mp 1)/2, \\ 0, & \text{otherwise} \end{cases}.$$

Let $\sigma_x(n)$ denote the sum of the x th powers of the divisors of n . In 1969, Lahiri [3] noticed that every definition of $\sigma_k(0) = f(n)$, $k \neq 1$ is irrelevant in order to keep the classical identity (2) and posed the following problem: "Whether analogous identities exist for divisor function $\sigma_k(n)$ of higher degree?" Formally, for every not necessarily integer value of x , $-\infty < x < \infty$, for $\sigma_x(n)$ we could consider an identity of type (2) of the form

$$\sigma(n) = g_x(n) + \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \dots,$$

where $\{g_x(n)\}$ is some "compensating sequence," and a solution of the Lahiri problem consists of a description of the compensating sequence for every n without a reference to its divisors. In particular, by the definition of $\sigma_x(n)$, and accepting as in (5) $\sigma(0) = n$, we find

$$\begin{aligned} g_x(1) &= 0, \quad g_x(2) = 2^x - 2, \quad g_x(3) = 3^x - 2^x - 1, \quad g_x(4) = 4^x - 3^x - 1, \\ g_x(5) &= 5^x - 4^x - 3^x - 2^x + 4, \quad g_x(6) = 6^x - 5^x - 4^x, \dots \end{aligned}$$

At first sight, this sequence is even more complicated than $\sigma_x(n)$, and it seems hardly probable to find a required description of it. Our paper, in particular, is devoted to this aim. For a simplification of our transformations, below we accept the unique convention

$$(8) \quad \sigma_x(n) = 0, \quad \text{if } n \leq 0$$

It is easy to see that in this case we have only a little change of the compensating sequence in the identity of the same form

$$\sigma_x(n) = h_x(n) + \sigma_x(n-1) + \sigma_x(n-2) - \sigma_x(n-5) - \sigma_x(n-7) + \dots,$$

such that

$$h_x(n) = g_x(n) + \begin{cases} (-1)^{m-1}, & \text{if } n = m(3m \mp 1)/2, \\ 0, & \text{otherwise} \end{cases}.$$

Note that this relation is so simple only due to Euler pentagonal identity (6); in more general case (see below Theorem 1) the corresponding relations could be very complicated and the convention (8) plays the unique role for the obtaining of general result. In particular, we write (1)-(2) in just a little another form. Namely, according to (8), instead of conventions (4)-(5), we accept the unique convention

$$p(0) = 0, \quad \sigma(0) = 0.$$

Then with help of (7) it is easy to see that, instead of (1)-(2), we have

$$p(n) = h^{(p)}(n) +$$

$$(9) \quad p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \dots,$$

where the compensating sequence has the form

$$(10) \quad h^{(p)}(n) = \begin{cases} (-1)^{m-1}, & \text{if } n = m(3m \mp 1)/2, \\ 0, & \text{otherwise} \end{cases}.$$

and, in view of the same structure of (1) and (2) and taking into account (4)-(5), we see that

$$\sigma(n) = h^{(\sigma)}(n) +$$

$$(11) \quad \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \sigma(n-12) + \sigma(n-15) - \dots,$$

where

$$(12) \quad h^{(\sigma)}(n) = \begin{cases} (-1)^{m-1}n, & \text{if } n = m(3m \mp 1)/2, \\ 0, & \text{otherwise} \end{cases}.$$

Before formulating a generalization of (9) and (11), we study the divisor function over divisors belonging to a prescribed finite sequence A of positive integers. In the trivial case of a one-element sequence $A = \{a\}$ we put

$$(13) \quad \sigma_x^{(\{a\})}(n) = \begin{cases} a^x, & \text{if } a|n, \quad n > 0, \\ 0, & \text{otherwise} \end{cases}, \quad x \in (-\infty, +\infty).$$

According to (13), we accept

$$(14) \quad \sigma_x^{(\{a\})}(n) = 0, \quad n \leq 0,$$

such that

$$(15) \quad \sigma_x^{(\{a\})}(n) = \sigma_x^{(\{a\})}(n-a) + \begin{cases} a^x, & \text{if } n = a, \\ 0, & \text{otherwise} \end{cases}.$$

Consider now, for a fixed $k \geq 1$, an arbitrary sequence

$$(16) \quad A = \{a_j\}_{j=1}^k$$

of positive integers. For a fixed x , let us consider an associated sequence

$$(17) \quad B(A; x) = \{b_i(x)\}_{i=1}^{2^k},$$

where

$$(18) \quad b_i(x) = a_{j_1}^x + a_{j_2}^x + a_{j_3}^x + \dots + a_{j_r}^x,$$

if the binary expansion of $i-1$ is

$$(19) \quad i-1 = 2^{j_1-1} + 2^{j_2-1} + \dots + 2^{j_r-1}, \quad 1 \leq j_1 < j_2 < \dots < j_r, \quad 1 \leq r \leq k.$$

In particular, since $2^k - 1 = 2^{1-1} + 2^{2-1} + \dots + 2^{k-1}$, then

$$(20) \quad b_{2^k}(x) = a_1^x + a_2^x + \dots + a_k^x,$$

while, since to $i = 1$ corresponds the empty set of terms in (19), then

$$(21) \quad b_1(x) = 0.$$

Furthermore,

$$(22) \quad b_2(x) = a_1^x, \quad b_3(x) = a_2^x, \quad b_4(x) = a_1^x + a_2^x, \quad \text{etc.}$$

Moreover, denote

$$(23) \quad b_i(1) = b_i, \quad 1 \leq i \leq 2^k.$$

For $n \geq 1$, consider divisor function over sequence A

$$(24) \quad \sigma_x^{(A)}(n) = \sum_{d|n, d \in A} d^x$$

in the understanding that every term d^x repeats correspondingly to the multiplicity of d in sequence A . Besides, we accept the convention

$$(25) \quad \sigma_x^{(A)}(n) = 0, \quad \text{if } n \leq 0.$$

Denote by $\{t_n\}$ the Thue-Morse sequence [4], [2] which is defined as

$$(26) \quad t_n = (-1)^{s(n)},$$

where $s(n)$ denotes the number of ones in the binary expansion of n .

Theorem 1. *In convention $\sigma(n \leq 0) = 0$, we have the following recursion*

$$(27) \quad \sigma_x^{(A)}(n) = h_x^{(A)}(n) + \sum_{i=2}^{2^k} t_{2i-1} \sigma_x^{(A)}(n - b_i)$$

where the compensating sequence $h_x^{(A)}(n)$ is defined as

$$(28) \quad h_x^{(A)}(n) = \sum_{i \geq 2: b_i = n} t_{2i-1} b_i(x).$$

Remark 1. *Taking into account that*

$$1 + s(i-1) = s(2(i-1) + 1) = s(2i-1),$$

we prefer to write t_{2i-1} instead of $-t_{i-1}$.

Note that, as follows from (28), for $n > b_{2^k}$, $h_x^{(A)}(n) = 0$ such that

$$(29) \quad \sigma_x^{(A)}(n) = \sum_{i=2}^{2^k} t_{2i-1} \sigma_x^{(A)}(n - b_i), \quad n > b_{2^k}.$$

Consider now the divisor function

$$(30) \quad \sigma_x(n) = \sum_{d|n} d^x.$$

Putting here

$$(31) \quad b_i(x) = j_1^x + j_2^x + j_3^x + \dots + j_r^x \quad (\text{and } b_i = b_i(1))$$

if the binary expansion of $i - 1$ is defined by (19), we obtain the following result.

Theorem 2. *We have*

$$(32) \quad \sigma_x(n) = h_x(n) + \sigma_x(n-1) + \sigma_x(n-2) - \sigma_x(n-5) - \sigma_x(n-7) + \sigma_x(n-12) + \sigma_x(n-15) - \dots,$$

where the compensating sequence $\{h_x(n)\}$ is defined as

$$(33) \quad h_x(n) = \sum_{i \geq 2: b_i=n} t_{2i-1} b_i(x), \quad n \geq 1.$$

Theorem 2 gives a solution of the Lahiri problem for divisor function $\sigma_x(n)$.

2. PROOF OF THEOREM 1

We use the induction over the number of elements of sequence A , the base of which is given by (15). Note that if, instead of $A = \{a_1, \dots, a_k\}$, to consider the sequence

$$(34) \quad A' = \{a_1, \dots, a_k, a_{k+1}\},$$

then we have

$$(35) \quad \sigma_x^{(A')}(n) = \sigma_x^{(A)}(n) + \sigma_x^{(\{a_{k+1}\})}(n).$$

Furthermore, in the case of A' , to every i , $1 \leq i \leq 2^k$, with the binary expansion (19) of $i - 1$ corresponds bijectively the number $2^k + i$ from $[2^k + 1, 2^{k+1}]$ with the expansion

$$2^k + i - 1 = 2^{j_1-1} + 2^{j_2-1} + \dots + 2^{j_r-1} + 2^k$$

such that the associated sequence has the form

$$(36) \quad b_i(x) = \begin{cases} a_{j_1}^x + a_{j_2}^x + a_{j_3}^x + \dots + a_{j_r}^x, & \text{if } 1 \leq i \leq 2^k, \\ a_{j_1}^x + a_{j_2}^x + a_{j_3}^x + \dots + a_{j_r}^x + a_{k+1}^x, & \text{if } 2^k + 1 \leq i \leq 2^{k+1}. \end{cases}$$

This means that, for $1 \leq l \leq 2^k$, we have

$$(37) \quad b_{l+2^k}(x) = b_l(x) + a_{k+1}^x \quad (\text{in particular, } b_{1+2^k}(x) = a_{k+1}^x).$$

Notice also, that

$$(38) \quad t_{2^{k+1}+1} = 1; \quad t_{2(l+2^k)-1} = t_{2l+2^{k+1}-1} = -t_{2l-1}$$

and

$$(39) \quad \sum_{1 \leq l \leq 2^k} t_{2l-1} = - \sum_{1 \leq l \leq 2^k} t_{l-1} = 0.$$

Suppose now that the theorem is true up to k . Then, using (37)-(38), we have

$$(40) \quad \begin{aligned} & \sum_{i=2}^{2^{k+1}} t_{2i-1} \sigma_x^{(A')}(n - b_i) = \\ & \sum_{i=2}^{2^k} t_{2i-1} \sigma_x^{(A')}(n - b_i) + \sum_{i=2^k+1}^{2^{k+1}} t_{2i-1} \sigma_x^{(A')}(n - b_i) = \\ & \sum_{i=2}^{2^k} t_{2i-1} \sigma_x^{(A')}(n - b_i) + \sum_{l=1}^{2^k} t_{2(l+2^k)-1} \sigma_x^{(A')}(n - b_{l+2^k}) = \\ & \sum_{i=2}^{2^k} t_{2i-1} \sigma_x^{(A')}(n - b_i) - \sum_{l=1}^{2^k} t_{2l-1} \sigma_x^{(A')}((n - b_l) - a_{k+1}). \end{aligned}$$

Furthermore, by (40) and (35), we have

$$(41) \quad \begin{aligned} & \sum_{i=2}^{2^{k+1}} t_{2i-1} \sigma_x^{(A')}(n - b_i) = \\ & \sum_{i=2}^{2^k} t_{2i-1} \sigma_x^{(A)}(n - b_i) + \sum_{i=2}^{2^k} t_{2i-1} \sigma_x^{\{a_{k+1}\}}(n - b_i) \\ & - \sum_{i=1}^{2^k} t_{2i-1} \sigma_x^{(A)}((n - b_i) - a_{k+1}) - \sum_{i=1}^{2^k} t_{2i-1} \sigma_x^{\{a_{k+1}\}}((n - b_i) - a_{k+1}). \end{aligned}$$

Note that, according to (15),

$$(42) \quad \begin{aligned} & \sum_{i=2}^{2^k} t_{2i-1} \sigma_x^{\{a_{k+1}\}}(n - b_i) - \sum_{i=1}^{2^k} t_{2i-1} \sigma_x^{\{a_{k+1}\}}((n - b_i) - a_{k+1}) \\ & = \sigma_x^{\{a_{k+1}\}}(n) + a_{k+1}^x \sum_{1 \leq i \leq 2^k: n-b_i=a_{k+1}} t_{2i-1}. \end{aligned}$$

Therefore, from (41) we find

$$\sum_{i=2}^{2^{k+1}} t_{2i-1} \sigma_x^{(A')}(n - b_i) = \sigma_x^{\{a_{k+1}\}}(n) + a_{k+1}^x \sum_{1 \leq i \leq 2^k: n-b_i=a_{k+1}} t_{2i-1} +$$

$$(43) \quad \sum_{i=2}^{2^k} t_{2i-1} \sigma_x^{(A)}(n - b_i) - \sum_{i=1}^{2^k} t_{2i-1} \sigma_x^{(A)}((n - a_{k+1}) - b_i),$$

or, using the inductive hypothesis, we have

$$(44) \quad \sum_{i=2}^{2^{k+1}} t_{2i-1} \sigma_x^{(A')}(n - b_i) = \sigma_x^{\{a_{k+1}\}}(n) + a_{k+1}^x \sum_{1 \leq i \leq 2^k: n-b_i=a_{k+1}} t_{2i-1} -$$

$$(\sigma_x^{(A)}((n - a_{k+1}) - h^{(A)}(n - a_{k+1})) + \sigma_x^{(A)}(n) - h^{(A)}(n)).$$

Furthermore,

$$(45) \quad \sum_{2 \leq i \leq 2^{k+1}: b_i=n} t_{2i-1} b_i(x) = \sum_{2 \leq i \leq 2^k: b_i=n} t_{2i-1} b_i(x) +$$

$$\sum_{2^{k+1} \leq i \leq 2^{k+1}: b_i=n} t_{2i-1} b_i(x) = h_x^{(A)} - \sum_{1 \leq l \leq 2^k: b_{l+2^k}=n} t_{2l-1} b_{l+2^k}(x) =$$

$$h_x^{(A)}(n) - h_x^{(A)}(n - a_{k+1}) - a_{k+1}^x \sum_{1 \leq l \leq 2^k: b_l=n-a_{k+1}} t_{2l-1}.$$

Finally, summing the results of (44) and (45), we complete our proof:

$$\sum_{i=2}^{2^{k+1}} t_{2i-1} \sigma_x^{(A')}(n - b_i) + \sum_{2 \leq i \leq 2^{k+1}: b_i=n} t_{2i-1} b_i(x) =$$

$$\sigma_x^{\{a_{k+1}\}}(n) + \sigma_x^{(A)}(n) = \sigma_x^{(A')}(n). \blacksquare$$

3. PROOF OF THEOREM 2

If to consider as a finite sequence A the sequence $A = A_k = \{1, 2, \dots, k\}$, then, for $n \leq k$, we have

$$(46) \quad \sigma_x^{(A_k)}(n) = \sigma_x(n)$$

and, by Theorem 1, the (\pm) -structure of $\sigma_x^{(A_k)}(n)$ is the same as in the case of $x = 1$ (see (11)). Therefore, independently from the summands (either $\sigma_1(n)$ or $\sigma_x(n)$) we have the same reductions, i.e.

$$(47) \quad \sigma_x(n) = \sigma_x^{(A_k)}(n) = h_x^{(A_k)}(n) +$$

$$\sigma_x^{(A_k)}(n - 1) + \sigma_x^{(A_k)}(n - 2) - \sigma_x^{(A_k)}(n - 5) - \sigma_x^{(A_k)}(n - 7) +$$

$$\sigma_x^{(A_k)}(n - 12) + \sigma_x^{(A_k)}(n - 15) - \dots, \quad (n \leq k),$$

with the compensating sequence

$$(48) \quad h_x^{(A_k)}(n) = \sum_{i \geq 2: b_i=n} t_{2i-1} b_i(x)$$

where $b_i(x)$ are defined by (31). If, instead of A_k , to consider N , then for every n we actually consider a finite part of (47) which corresponds to $A_n = \{1, 2, \dots, n\}$. Thus (47) is true for $A = N$, and (32)-(33) follow. ■

Example 1. Consider the case of $x = 1$, i.e. the case of sum-of-divisors function.

Then we have

$$h_1^{(N)}(n) = n \sum_{i \geq 2: b_i = n} t_{2i-1} = -n \sum_{i \geq 2: b_i = n} (-1)^{s(i-1)} = n(p_o(n) - p_e(n))$$

and, in view of (7), we obtain (11) as a special case of Theorem 2.

4. EXPRESSION OF COMPENSATING SEQUENCE $\{h_x(n)\} = \{h_x(n)^{(N)}\}$ VIA KNOWN SEQUENCES

Note that from the definition of sequence $b_n(x)$ (see (31) and (19)) it follows that if

$$(49) \quad n - 1 = \sum_{i \geq 1} \beta(i) 2^{i-1}$$

is the binary expansion of $n - 1$, then

$$(50) \quad b_n(x) = \sum_{i \geq 1} \beta(i) i^x,$$

such that

$$(51) \quad b_n = \sum_{i \geq 1} \beta(i) i.$$

Notice that, (51) is Sequence A029931(n-1) in [6]. Denoting

$$(52) \quad A029931(n) = \eta(n),$$

according to (33), we have

$$(53) \quad h_x(n) = \sum_{j \geq 1: \eta(j) = n} (-1)^{s(j)-1} b_{j+1}(x).$$

Example 2. Consider the case of $x = 0$, i.e. the case of the number of divisors of n .

Then, by (50) and (53), the compensating sequence has the form

$$(54) \quad h_0(n) = \sum_{j \geq 1: \eta(j) = n} (-1)^{s(j)-1} s(j),$$

where $s(n)$, as in the above, is the number of ones in the binary expansion of n . The first terms of compensating sequence $\{h_0^{(N)}(n)\}_{n \geq 1}$ are:

$$(55) \quad 1, 1, -1, -1, -3, 0, -2, 1, 2, 1, 2, 4, 1, -1, \dots$$

Note that, in view of (49), the expression (51) gives the number of all partitions with distinct parts of a fixed values of b_n . This means that if to denote by e_n the set of the terms of A029931 for which $\eta(j) = n$:

$$e_1 = \{1\}, e_2 = \{2\}, e_3 = \{3, 4\}, e_4 = \{5, 8\}, e_5 = \{6, 9, 16\}, \\ e_6 = \{7, 10, 17, 32\}, e_7 = \{11, 12, 18, 33, 64\} \dots,$$

then the concatenation of this sets leads to the ordering of all partitions of n with distinct parts :

$$n = \sum_{i \geq 1} \beta(i) i$$

respectively to the values of $\sum_{i \geq 1} \beta(i) 2^{i-1}$. Thus this way leads us to the Adams-Watters sequence "Decimal equivalent of binary encoding of partitions into distinct parts" (see A118462 in [6]):

$$(56) \quad 1, 2, 3, 4, 5, 8, 6, 9, 16, 7, 10, 17, 32, 11, 12, 18, 33, 64, \dots$$

Denote Sequence (56) via $W(n)$ and put $|e_n| = R(n)$, where, for $n \geq 1$, $\{R(n)\}$ is Sequence A000009[6] that is the number of partitions of n into distinct parts. Finally, denote

$$(57) \quad T(n) = \sum_{k=1}^n R(k) = A036469(n) - 1, \quad n \geq 1.$$

Then from (53) we find

$$(58) \quad h_x(n) = \sum_{m=0}^{R(n)-1} (-1)^{s(W(T(n)-m))} b_{W(T(n)-m)+1}(x).$$

Example 3. Let us calculate the seventh term $h_0(7)$ of sequence (55).

By (58), here we have

$$(59) \quad h_0(n) = \sum_{m=0}^{R(n)-1} (-1)^{s(W(T(n)-m))} s(W(T(n)-m)).$$

If $n = 7$, then we have from the corresponding tables of [6]:

$$R(n) = 5, \quad T(n) = 18, \quad W(18) = 64, \quad W(17) = 33, \\ W(16) = 18, \quad W(15) = 12, \quad W(14) = 11.$$

Thus, according to (59), we find

$$h_0(7) = 1 - 2 - 2 - 2 + 3 = -2.$$

5. SOME ANOTHER IDENTITIES

In case of the finite set

$$A_k = \{2^{j-1}\}_{j=1}^k,$$

according to (18), we have

$$(60) \quad b_i(x) = 2^{(j_1-1)x} + 2^{(j_2-1)x} + \dots + 2^{(j_r-1)x},$$

if

$$(61) \quad i-1 = 2^{j_1-1} + 2^{j_2-1} + \dots + 2^{j_r-1}, \quad 1 \leq j_1 < j_2 < \dots < j_r, \quad 1 \leq r \leq k.$$

Thus b_i has an especially simple form:

$$(62) \quad b_i = i-1.$$

Let

$$n = 2^{\alpha_1-1} + \dots + 2^{\alpha_m-1}.$$

Then, according to Theorem 1 and (62), we find

$$(63) \quad h_x^{(A_k)}(n) = \sum_{i \geq 2: b_i=n} t_{2i-1} b_i(x) = t_{2n+1} b_{n+1}(x) = (-1)^{s(2n+1)} (2^{(\alpha_1-1)x} + \dots + 2^{(\alpha_m-1)x})$$

and

$$(64) \quad \sigma_x^{(A_k)}(n) = (-1)^{s(2n+1)} (2^{(\alpha_1-1)x} + \dots + 2^{(\alpha_m-1)x}) + \sum_{i \geq 2} (-1)^{s(2i-1)} \sigma_x^{(A_k)}(n - (i-1)).$$

Considering now the infinite sequence of powers of 2:

$$A = \{2^{j-1}\}_{j \geq 1},$$

we conclude that

$$\sigma_x^{(A)}(n) = (-1)^{s(2n+1)} (2^{(\alpha_1-1)x} + \dots + 2^{(\alpha_m-1)x}) + \sum_{i \geq 2}^n (-1)^{s(2i-1)} \sigma_x^{(A)}(n - (i-1)),$$

or

$$(65) \quad \sigma_x^{(A)}(n) = (-1)^{s(2n+1)} (2^{(\alpha_1-1)x} + \dots + 2^{(\alpha_m-1)x}) - \sum_{j=1}^{n-1} (-1)^{s(n-j)} \sigma_x^{(A)}(j),$$

where

$$n = 2^{\alpha_1-1} + \dots + 2^{\alpha_m-1}.$$

In particular, in the case of $x = 0$, we obtain the identity

$$(66) \quad \sum_{j=1}^n (-1)^{s(n-j)} \sigma_0^{(A)}(j) = (-1)^{s(n)-1} s(n).$$

The sequence $\{\sigma_0^{(A)}(n) - 1\}_{n \geq 1}$ is well-known so-called "the binary carry sequence" (A007814 in [6]). In the case of $x = 1$, we obtain the identity

$$(67) \quad \sum_{j=1}^n (-1)^{s(n-j)} \sigma_1^{(A)}(j) = (-1)^{s(n)-1} n.$$

The sequence $\{\sigma_1^{(A)}(n)\}_{n \geq 1}$ is also well-known (see A038712 in [6]).

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